Vectoranalyse

Solutions Midtoets May 2012

1. (a) The point where the two given surfaces touch (or intersect) is given by

$$x^2 + 3x + y^2 + 2y = -x^2 + 3x - y^2 + 2y.$$

It follows that in this point $x^2 + y^2 = 0$ holds, which can be satisfied only if $x = y = 0$. Consequently z has to be zero as well.
To find the tangent plane we look at

$$\nabla \left(\pm x^2 + 3x \pm y^2 + 2y - z\right) = (\pm 2x + 3, \pm 2y + 2, -1).$$

We see that the normal vector of both surfaces is equal to $(3, 2, -1)$ if $x = y = z = 0$. The tangent plane U is everything orthogonal to the normal vector, in other words it is given by

$$(3, 2, -1)\,(x, y, z) = 0.$$

(b) We are looking for two tangent planes touching the unit sphere, that are parallel to U. From (a) we know that U is normal to the vector $(3, 2, -1)$. We thus need to find two points on the sphere where the tangent planes are normal to the vector $(3, 2, -1)$. We know that for each point on the sphere the position vector of this point is orthogonal to the tangent plane of the sphere at this point. Hence we get the two desired points on the sphere by normalizing $(3, 2, -1)$ and taking into account the possible directions of the resulting vector. The points on the sphere are thus given by by $\pm\frac{(3,2,-1)}{\sqrt{14}}$.

2. (a) If $z(x, y)$ is considered to be implicitly defined through $F(x, y, z) = 0$ on a neighborhood of some (x_0, y_0, z_0), to obtain the partial derivatives we differentiate as follows:

$$0 = \frac{\partial}{\partial x} F(x, y, z(x, y)) = \frac{\partial F(x, y, z(x, y))}{\partial x} + \frac{\partial F(x, y, z(x, y))}{\partial z}\frac{\partial z(x, y)}{\partial x}.$$

We see that

$$\frac{\partial z(x, y)}{\partial x} = -\frac{\frac{\partial F(x,y,z(x,y))}{\partial x}}{\frac{\partial F(x,y,z(x,y))}{\partial z}}. \tag{1}$$

Similarly for y.

(b) For $F(x, y, z) = xyz - 2 = 0$ these partial derivatives are

$$\frac{\partial z(x, y)}{\partial x} = -\frac{z}{x}, \quad \frac{\partial z(x, y)}{\partial y} = -\frac{z}{y}.$$

(c) Using (1) it is obvious that

$$\left(\frac{\partial x(y, z)}{\partial y}\right)\left(\frac{\partial y(x, z)}{\partial z}\right)\left(\frac{\partial z(x, y)}{\partial x}\right) = \left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)\left(-\frac{F_x}{F_z}\right) = -1.$$

3. Consider $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 3x^2 + 2y^2 + z^2 - 1$.

We are looking for the point on the ellipsoid that is closest to the origin, thus a minimum of f satisfying $g = 0$. Define

$$L(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(3x^2 + 2y^2 + z^2 - 1)$$

and recall that every critical point of L is a critical point of f on the set $\{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$. We find that

$$\begin{aligned}
\frac{\partial L}{\partial x} &= (6\lambda + 2)x = 0, \\
\frac{\partial L}{\partial y} &= (4\lambda + 2)y = 0, \\
\frac{\partial L}{\partial z} &= (2\lambda + 2)z = 0, \\
\frac{\partial L}{\partial \lambda} &= 3x^2 + 2y^2 + z^2 - 1 = 0.
\end{aligned}$$

There are three critical points:

$$\begin{aligned}
&\left(\tfrac{1}{\sqrt{3}}, 0, 0\right) \quad \text{if} \quad \lambda = -\frac{1}{3}, \\
&\left(0, \tfrac{1}{\sqrt{2}}, 0\right) \quad \text{if} \quad \lambda = -\frac{1}{2}, \\
&(0, 0, 1) \quad \text{if} \quad \lambda = -1.
\end{aligned}$$

For $\lambda = -\frac{1}{3}$ the Hessian matrix is

$$\begin{pmatrix} 0 & 2\sqrt{3} & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{3} \end{pmatrix}.$$

We see that $\det H_3 < 0$ and $\det H_4 < 0$. This is equivalent to $-d_3 > 0$ and $-d_4 > 0$, which means that $\left(\frac{1}{\sqrt{3}}, 0, 0\right)$ is a minimum. We conclude that the largest sphere centered at the origin that lies completely in the given ellipsoid has a radius of $\frac{1}{\sqrt{3}}$.

(b) Similarly we find that $\left(0, \frac{1}{\sqrt{2}}, 0\right)$ is a saddle point and $(0, 0, 1)$ is a maximum. The smallest sphere centered at the origin that lies completely outside of the given ellipsoid is

$$x^2 + y^2 + z^2 = 1.$$